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ON SPECTRUM ASSIGNMENT OF INFINITE-DIMENSIONAL LINEAR SYSTEMS BY BOUNDED LINEAR FEEDBACK

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On Spectrum Assignment Of Infinite-Dimensional Linear Systems By Bounded Linear Feedback

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ABSTRACT : This paper deals with the spectrum assignability via bounded linear feedback. The necessary and sufficient condition of Sun [21] is generalized to a class of boundary control systems. A natural application of our theoretical results is briefly presented. The results obtained in this paper have potential applications in non-dissipative spectral systems.

KEY WORDS : Distributed parameter systems, bounded linear feedback control, perturbation, spectral determination, stability, flexible rotating structures.

Sur Le Placement du Spectre des Systèmes de Dimension Infinie Par Feedback Linénaire Borné

RESUME : Cet article étudie le placement du spectre pour des systèmes à parameters répartis par feedback linénaire borné. La condition nécessaire et suffisante de Sun [21] est généralisée pour une classe de systèmes de contrôle à la frontière. Une application naturelle de nos résultats théoriques est brièvement présentée. Les résultats obtenus dans cet article auront de prospective applications pour des systèmes spectraux non-dissipatifs.

MOTS CLES : Systèmes à paramètres répartis, feedback linéaire borné, perturbation, détermination spectrale, stabilité, structures flexibles en rotation.

1 INTRODUCTION

In this paper, we consider directly the following infinite-dimensional linear systems on a separable Hilbert space H (the inner product and induced norm in H are denoted respectively by $< \cdot, \cdot >$ and $\| \cdot \|$)

$$(\Sigma_O) \ : \qquad \dot{X}(t) = AX(t) + bu(t)$$

where A is an infinitesimal generator of C_0-semigroup on H and the input element b is not necessarily admissible in the sense of [11]. Throughout the paper, A and b are assumed to satisfy the conditions H1, H2 and H3.

HYPOTHESIS H1 : A is a regular spectral operator. We suppose for simplicity of exposition that the spectrum $\sigma(A) = \{\lambda_n, n \in \mathbb{N}\}$ is simple.

HYPOTHESIS H2 : $\mathcal{D}(A^*)$ is the domain of the adjoint operator A^*. $\mathcal{D}'(A^*)$ is the topological dual of $\mathcal{D}(A^*)$. We suppose that b belongs to $\mathcal{D}'(A^*)$.

HYPOTHESIS H3 : The eigenvectors $\{\phi_k; k \in \mathbb{N}\}$ of A form a Riesz basis in H. The dual basis corresponding to the eigenvectors of A^* is denoted by $\{\psi_k; k \in \mathbb{N}\}$. We set $b_k = (\psi_k, b)$ where $(\cdot, \cdot)$ is the classical duality product on $\mathcal{D}(A^*) \times \mathcal{D}'(A^*)$. It can be defined as the continuous extension of the inner product on H (H is dense in $\mathcal{D}'(A^*)$). d_n is the distance of λ_n to the rest of the spectrum $\sigma(A)$. Again to simplify, we suppose that $b_k \neq 0$ for all $k \in \mathbb{N}$. Now suppose that there exists a constant $0 \leq \tau < 1$ such that

$$(i) \ : \qquad \sum_{n=1}^{+\infty} \left(\frac{|b_n|}{d_n^\tau} \right)^2 < +\infty$$

and

$$(ii) \ : \qquad \sum_{n=1}^{+\infty} \left(\frac{1}{d_n} \right)^{3(1-\tau)} < +\infty.$$

The assumptions H2 and H3 allow to take into account the cases of boundary control because many linear P.D.E.'s may be formulated in the form of (Σ_O) (see [11]). In particular, the case of [21], the cantilever beam model with lateral force control [17] [18] and heat conduction equations [11] enter the class of the systems considered here. However the cantilever beam model with moment force control gives rise to a distributed parameter system of the form (Σ_O) with non-admissible input.

Sun proved in [21] that under the hypothesis H1 with $b \in H$ and a stronger hypothesis than H3 (ii) (hypothesis H3 (i) is automatically verified when $b \in H$ with $\tau = 0$), the necessary and sufficient condition for the operator $A + b < \cdot, h >$ ($h \in H$) to have the spectrum $\{v_k; k \in \mathbb{N}\}$ assigned is :

$$(SA) \; : \qquad \sum_{k\in\mathbb{N}} \left| \frac{v_k - \lambda_k}{b_k} \right|^2 < +\infty$$

Then more results on spectrum assignment via linear feedback at the boundary have been obtained in [10] and [18]. Especially, Rebarber has shown that for some cases, it is possible to assign uniformly an infinite number of eigenvalues by unbounded but "admissible" linear feedback at the boundary [17] [18].

In this paper, we do not restrict to the case of admissible input elements [10], but consider input elements in $\mathcal{D}'(A^*)$. We restrict to the case of bounded linear feedbacks (abbreviated as BLF) : $u(t) =< x(t), h >$ with $h \in H$. We prove that under the hypothesis H1, H2 and H3, the condition (SA) is also a necessary and sufficient condition for the spectrum assignment. We give also explicitly the BLF which realizes the spectrum assignment. We deduce also that with spectral assignment we have the spectrum determined growth assumption for the closed loop system semigroup. Our results allow to explain why the system (Σ_O) cannot be exponentially stabilized by the BLF laws when it has an infinite number of eigenvalues in $\Re e(s) \geq 0$ and its input vector b is admissible in sense of [11]. However we prove that in some cases, the uniform assignment of the spectrum can be achieved by BLF. Our demonstration is made using essentially the perturbation theory of regular spectral operators of [14] [7]. We should point out that the advantage of using BLF's is that the construction of BLF laws is simple and systematic as illustrated by our example.

The paper is organized as follows : Section 2 is devoted to the spectral characterization of the closed loop operator. In Section 3, we give explicitly the BLF which realizes the spectrum assignment and discuss the stabilization limitation with BLF. Finally we point out some possible developements of our results.

2 PERTURBATION RESULTS

As only BLF laws are considered in the paper, the closed loop system is governed by the evolution equation (Σ_C) in the phase space H :

$$(\Sigma_C) : \qquad \dot{X}(t) = AX(t) + b < X(t), h >$$

The linear operator $A : \mathcal{D}(A) \longrightarrow H \subset \mathcal{D}'(A^*)$ admits the unique extension $\hat{A} \in \mathcal{L}(H, \mathcal{D}'(A^*))$ by continuity because $\mathcal{D}(A)$ is dense in H. Accordingly, the linear operator $A_h = A + b < \cdot, h >: \mathcal{D}(A) \rightarrow \mathcal{D}'(A^*)$ admits a unique extension from H to $\mathcal{D}'(A^*)$ still noted by A_h and for all $x \in H$, $A_h x = \hat{A}x + b < x, h >$.

Define now $\mathcal{D}(A_h) = \{x \in H; \hat{A}x + b < x, h > \in H\}$. We use here the same definition for the unbounded linear operator $A_h : \mathcal{D}(A_h) \longrightarrow H$ as that of [18]. In the following , instead of directly dealing with A_h, we study the unbounded linear operator $L_h = A^* + h(\cdot, b)$ because the infinitesimal generation property is equivalent between A_h and L_h. It is easy to see that $\mathcal{D}(L_h) = \mathcal{D}(A^*)$ from the hypothesis H2 and that L_h is closed because $h(\cdot, b)$ is A^*-compact ([6], p.194).

The main result of this section is to prove that L_h considered as perturbation of A^* is regular spectral, and that the eigenvalues of L_h can be located in the discs centered at the eigenvalues of A^* (Theorem 1). Moreover we show that the corresponding eigenvectors of L_h form a Riesz basis in H. In Theorem 2 we generalize a result of Sun [21] by refining the radius of the discs of Theorem 1.

LEMMA 1 : *The unbounded linear operator $A_h : \mathcal{D}(A_h) \longrightarrow H$ is the adjoint operator of L_h with the inner product $< \cdot, \cdot >$ on H.*

<u>Proof</u> : For all $x \in \mathcal{D}(L_h^*)$ and $y \in \mathcal{D}(L_h^2)$, one has :

$$(y, A_h x) = (y, \hat{A}x + b < x, h >) = (y, \hat{A}x) + < h, x > (y, b) = (y, \hat{A}x) + < h(y, b), x > . \tag{1}$$

The dual operator $\hat{A}'$ on $\mathcal{D}(A^*)$ satisfies $\hat{A}'y = A^*y$ for all $y \in \mathcal{D}(L_h^2)$. Indeed for all $x \in \mathcal{D}(A)$ and $y \in \mathcal{D}(L_h^2)$,

$$(y, \hat{A}x) = (y, Ax) = < y, Ax > = < A^*y, x > = (A^*y, x)$$

Since $\mathcal{D}(A)$ is dense in $\mathcal{D}'(A^*)$, $y \in \mathcal{D}(\hat{A}')$ and $\hat{A}'y = A^*y$. Thus from (1), one obtains that

$$(y, A_h x) = (A^*y + h(y, b), x) = (L_h y, x) = < L_h y, x > = < y, L_h^* x > = (y, L_h^* x). \tag{2}$$

$\mathcal{D}(L_h^2)$ being dense in $\mathcal{D}(A^*)$, $A_h x = L_h^* x$ in $\mathcal{D}'(A^*)$. This means that $A_h \supset L_h^*$.

On the other hand, for all $x \in \mathcal{D}(A_h)$ and all $y \in \mathcal{D}(L_h^2)$, using the first part of (2) we have $< y, A_h x > = (y, A_h x) = < L_h y, x >$. This means exactly that $x \in \mathcal{D}(L_h)$ and $A_h x = L_h^* x$. Hence $A_h \subset L_h^*$. Therefore $A_h = L_h^*$. □

The following elementary inequalities will be used systematically in order to make the majorations considered later.

LEMMA 2 : *Define the circles $C_n = \left\{\lambda; |\lambda - \overline{\lambda}_n| = \frac{1}{3}d_n\right\}$ and the discs $D_n = \left\{\lambda; |\lambda - \overline{\lambda}_n| \le \frac{1}{3}d_n\right\}$. Under the H1, H2 and H3, $\lambda \in C_n$ implies that for all $x \in H$,*

$$|(R(\lambda; A^*)x, b)| \le K\|x\| \left(\frac{1}{d_n}\right)^{1-\tau} \quad \textit{for some positive constant } K;$$

($R(\lambda;\cdot) = (\lambda I - \cdot)^{-1}$) and $\lambda \in D_n$ implies that for all $x = \sum_{i=1}^{+\infty} x_i\psi_i \in H$,

$$\left| \sum_{i\in\mathbb{N},\, i\neq n}^{+\infty} \frac{x_i b_i}{\lambda - \overline{\lambda}_i} \right| \leq K \left(\frac{1}{d_n}\right)^{1-\tau} \|x\|.$$

<u>Proof</u> : Since $\{\psi_i\}_{i\in\mathbb{N}}$ is a Riesz basis of H, each $x \in H$ has a unique Fourier expansion : $x = \sum_{i=1}^{+\infty} x_i\psi_i$ and there exist positive constants a_1 and a_2 such that $a_1^2 \sum_{i=1}^{+\infty} |x_i|^2 \leq \|x\|^2 \leq a_2^2 \sum_{i=1}^{+\infty} |x_i|^2$ (see [9]). From H2, for all $\lambda \in \rho(A^*)$

$$(R(\lambda; A^*)x, b) = \sum_{i=1}^{+\infty} \frac{x_i b_i}{\lambda - \overline{\lambda}_i}. \tag{3}$$

We have to note that $\bigcap_{n\in\mathbb{N}} C_n = \emptyset$. Hence, $\lambda \in C_n$ implies that $|\lambda - \lambda_i| \geq \frac{1}{3}d_i$ for all $i \in \mathbb{N}$. Moreover for all $\lambda \in C_n$, we have for $i \neq n$:

$$|\lambda - \lambda_i| \geq |\lambda_n - \lambda_i| - |\lambda - \lambda_n| \geq |\lambda_n - \lambda_i| \left(1 - \frac{|\lambda - \lambda_n|}{|\lambda_n - \lambda_i|}\right)$$

$$\geq |\lambda_n - \lambda_i| \left(1 - \frac{\frac{1}{3}d_n}{d_n}\right) \geq \frac{2}{3}d_n. \tag{4}$$

Applying the Cauchy inequality and (4) in (3) and then using H3 (i) and (ii) we get the following for all $\lambda \in C_n$:

$$|(R(\lambda; A^*)x, b)| \leq \left(\sum_{i=1}^{+\infty} |x_i|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{+\infty} \left|\frac{b_i}{\lambda - \overline{\lambda}_i}\right|^2\right)^{\frac{1}{2}}$$

$$= \left(\sum_{i=1}^{+\infty} |x_i|^2\right)^{\frac{1}{2}} \left[\sum_{i=1}^{+\infty} \frac{|b_i|^2}{|\lambda - \overline{\lambda}_i|^{2\tau}} \left(\frac{1}{|\lambda - \overline{\lambda}_i|}\right)^{2(1-\tau)}\right]^{\frac{1}{2}} \leq \frac{\|x\|}{a_1} \left[\sum_{i=1}^{+\infty} \frac{|b_i|^2}{(\frac{d_i}{3})^{2\tau}}\right]^{\frac{1}{2}} \left(\frac{3}{d_n}\right)^{1-\tau}$$

$$\leq \frac{\|x\|}{a_1} \tilde{M} \left(\frac{1}{d_n}\right)^{1-\tau}.$$

This proves the first inequality.

Using (4), it is easy to see that for all $\lambda \in D_n$, $\mu \in C_n$ and $i \neq n$,

$$|\lambda - \overline{\lambda}_i| \geq |\mu - \overline{\lambda}_i| - |\lambda - \mu| \geq |\mu - \overline{\lambda}_i| \left(1 - \frac{|\lambda - \mu|}{|\mu - \overline{\lambda}_i|}\right) \geq |\mu - \overline{\lambda}_i| \left(1 - \frac{\frac{1}{3}d_n}{\frac{2}{3}d_n}\right) \geq \frac{1}{2}|\mu - \overline{\lambda}_i|.$$

Then the same argument as above can be used to prove the second inequality. □

Let us define the complex functions $\Delta_h(\lambda) = (R(\lambda; A^*)h, b) - 1$ for $\lambda \in \rho(A^*)$, which will be useful later on. It was shown in [10] that $\Delta_h(\lambda)$ is analytic on $\rho(A^*)$. So consider the zero set of this analytic function denoted by $\Lambda = \{\lambda \in \rho(A^*); \Delta_h(\lambda) = 0\}$. Set $T = h(\cdot, b)$ such that for all $x \in \mathcal{D}(A^*)$, $Tx = h(x, b)$. From H2 $\mathcal{D}(T) \supset Range(R(\lambda; A^*))$ for all $\lambda \in \rho(A^*)$, and therefore $TR(\lambda; A^*) \in \mathcal{L}(H, H)$ (see [7]). By direct computations we can prove that for $\lambda \notin \Lambda \bigcup \sigma(A^*)$,

$$R(\lambda; L_h) = R(\lambda; A^*) + R(\lambda; A^*)TR(\lambda; A^*) +$$

$$(R(\lambda; A^*)h, b)R(\lambda; A^*)TR(\lambda; A^*) + \frac{(R(\lambda; A^*)h, b)^2 R(\lambda; A^*)TR(\lambda; A^*)}{1 - (R(\lambda; A^*)h, b)}. \qquad (5)$$

$R(\lambda; A^*)$ being compact, it is implied from (5) that $R(\lambda; L_h)$ is compact and so L_h is regular [14] [7]. Consequently, L_h has only a finite number of spectrum points on each compact of the complex plane.

For the convenience of the reader, we prove the following result (see also [10]) :

PROPOSITION 1 : *For each $0 \neq h \in H$, the discrete spectrum $\sigma(L_h)$ of the closed loop system is characterized by :*

$$\sigma(L_h) = \Lambda \bigcup \{\overline{\lambda}_n; < \phi_n, h >= 0\}.$$

<u>Proof</u> : From (5), $\lambda \notin \Lambda \bigcup \sigma(A^*)$ implies that $\lambda \in \rho(L_h)$. It follows that $\sigma(L_h) \subset \Lambda \bigcup \sigma(A^*)$. Moreover one can also show (Proposition 4.1 of [10]) that as soon as $< \phi_n, h > \neq 0$ and $b_n =< \psi_n, b > \neq 0$, $\overline{\lambda}_n \in \rho(L_h)$. Hence $\sigma(L_h) \subset \Lambda \bigcup \{\overline{\lambda}_n; < \phi_n, h >= 0\}$. Now let $\tilde{\lambda} \in \Lambda$. By direct computation, one can prove that $(\tilde{\lambda} - L_h)R(\tilde{\lambda}; A^*)h = 0$. Because of $R(\tilde{\lambda}; A^*)h \neq 0$, $\tilde{\lambda} \in \sigma(L_h)$. Let $\overline{\lambda}_m \in \sigma(A^*)$ such that $< \phi_m, h >= 0$. Then $A_h\phi_m = [\hat{A} + b < \cdot, h >]\phi_m = \lambda_m\phi_m$. Therefore $\phi_m \in \mathcal{D}(A_h)$ and $\lambda_m \in \sigma(A_h)$. From Lemma 1, it is true that $\overline{\lambda}_m \in \sigma(L_h)$ (see p. 184,[6]). This means that $\sigma(L_h) \supset \Lambda \bigcup \{\overline{\lambda}_n; < \phi_n, h >= 0\}$. Consequently, $\sigma(L_h) = \Lambda \bigcup \{\overline{\lambda}_n; < \phi_n, h >= 0\}$. □

In fact, more can be said on the spectrum $\sigma(L_h)$ as follows :

THEOREM 1 : *Under H1, H2 and H3, we have :*

a) for all but a finite number of n, each element $\overline{v}_n$ of the spectrum $\sigma(L_h)$ is in the corresponding disc D_n;

b) L_h is regular spectral and the corresponding spectral mesures satisfy :

$$\sum_{\lambda \in \sigma(L_h)} E'_\lambda = I.$$

Proof : The proof of a) is contained in that of b). So we prove directly b). From H3 (i) we have for all $\lambda \notin \bigcup_{n\geq 1} D_n \bigcup\{\lambda; |\lambda| \leq M1\}$:

$$|(R(\lambda; A^*)h, b)| = |\sum_{k=1}^{+\infty} \frac{h_k b_k}{\lambda - \overline{\lambda}_k}| \leq \sum_{k=1}^{+\infty} \frac{|h_k b_k|}{|\lambda - \overline{\lambda}_k|^{\tau} |\lambda - \overline{\lambda}_k|^{1-\tau}}$$

$$\leq \sum_{k=1}^{+\infty} \frac{|h_k b_k|}{|\lambda - \overline{\lambda}_k|^{\tau}} \sup_{k\in\mathbb{N}} \frac{1}{|\lambda - \overline{\lambda}_k|^{1-\tau}} \leq K\|h\| \sup_{k\in\mathbb{N}} \frac{1}{|\lambda - \overline{\lambda}_k|^{1-\tau}}. \tag{6}$$

From H3 (ii) there exists a positive integer N_1 such that for all $k \geq N_1$, $K\|h\| \left(\frac{3}{d_k}\right)^{1-\tau} \leq \frac{1}{2}$. We can also choose M_1 sufficiently large in the above such that for all $1 \leq k \leq N_1$, $|\lambda - \overline{\lambda}_k| \geq \frac{1}{3} d_{N_1}$. Therefore there is some M_1 large enough such that $(R(\lambda; A^*)h, b) \leq \frac{1}{2}$ and $\lambda \in \rho(L_h)$ for all $\lambda \notin \bigcup_{n\geq 1} D_n \bigcup\{\lambda; |\lambda| \leq M1\}$. In other words, all spectrum points $\sigma(L_h)$ but a finite number of them are recovered by the union of D_n's. Then we define in the following the spectral measures associated respectively with the spectrum points of A^* and L_h but for a finite number of them :

$$E(\lambda_n) = \frac{1}{2\pi i} \oint_{C_n} R(\lambda; A^*) d\lambda \text{ and } E'_n = \frac{1}{2\pi i} \oint_{C_n} R(\lambda; L_h) d\lambda. \tag{7}$$

From H3, $\sum_{n=1}^{+\infty} E(\lambda_n) = I$ where the infinite sum converges stronly in $\mathcal{L}(H,H)$.

For L_h to be spectral, it is sufficient to show that the following expression (where N_1 will be determined in the context) is uniformly bounded in $\mathcal{L}(H,H)$ with respect to N_2 :

$$[\sum_{n\geq N_1}^{N_2} E'_n - E(\lambda_n)] = \sum_{n\geq N_1}^{N_2} \frac{1}{2\pi i} \oint_{C_n} R(\lambda; A^*) T R(\lambda; A^*) d\lambda$$

$$+ \sum_{n\geq N_1}^{N_2} \frac{1}{2\pi i} \oint_{C_n} (R(\lambda; A^*)h, b) R(\lambda; A^*) T R(\lambda; A^*) d\lambda$$

$$+ \sum_{n\geq N_1}^{N_2} \frac{1}{2\pi i} \oint_{C_n} \frac{(R(\lambda; A^*)h, b)^2 R(\lambda; A^*) T R(\lambda; A^*)}{1 - (R(\lambda; A^*)h, b)} d\lambda. \tag{8}$$

It is easy to find that for all $x = \sum_{k=1}^{+\infty} x_k \psi_k$,

$$R(\lambda; A^*) T R(\lambda; A^*) x = \left[\sum_{j\neq n} \frac{x_j b_j}{\lambda - \overline{\lambda}_j}\right] \left[\sum_{j\neq n} \frac{h_j \psi_j}{\lambda - \overline{\lambda}_j}\right] + \frac{x_n b_n h_n \psi_n}{(\lambda - \overline{\lambda}_n)^2}$$

$$+ \frac{x_n b_n}{\lambda - \overline{\lambda}_n} \sum_{j\neq n} \frac{h_j \psi_j}{\lambda - \overline{\lambda}_j} + \frac{h_n \psi_n}{\lambda - \overline{\lambda}_n} \sum_{j\neq n} \frac{x_j b_j}{\lambda - \overline{\lambda}_j}.$$

By virtue of Lemma 2, we observe that that $\sum_{j\neq n}\frac{h_j\psi_j}{\lambda-\overline{\lambda}_j}$ and $\sum_{j\neq n}\frac{x_jb_j}{\lambda-\overline{\lambda}_j}$ are analytic in the disc D_n. Thus using the Residue Theorem, we get :

$$\frac{1}{2\pi i}\oint_{C_n} R(\lambda;A^*)TR(\lambda;A^*)xd\lambda = x_nb_n\sum_{j\neq n}\frac{h_j\psi_j}{\overline{\lambda}_n-\overline{\lambda}_j}+h_n\psi_n\sum_{j\neq n}\frac{x_jb_j}{\overline{\lambda}_n-\overline{\lambda}_j}. \tag{9}$$

and

$$\sum_{n\geq N_1}^{N_2}\frac{1}{2\pi i}\oint_{C_n} R(\lambda;A^*)TR(\lambda;A^*)xd\lambda = \sum_{n\geq N_1}^{N_2}x_nb_n\sum_{j\neq n}\frac{h_j\psi_j}{\overline{\lambda}_n-\overline{\lambda}_j}+\sum_{n\geq N_1}^{N_2}h_n\psi_n\sum_{j\neq n}\frac{x_jb_j}{\overline{\lambda}_n-\overline{\lambda}_j}.$$

Let us fix the constant $\tilde{M}$ once for all which will be clearly determined in the context (and which will be independent of N_2). Using Lemma 2, it is not difficult to obtain the following estimation :

$$\|\sum_{n\geq N_1}^{N_2}\frac{1}{2\pi i}\oint_{C_n} R(\lambda;A^*)TR(\lambda;A^*)xd\lambda\|$$

$$\leq \|\sum_{n\geq N_1}^{N_2}x_nb_n\sum_{j\neq n}\frac{h_j\psi_j}{\overline{\lambda}_n-\overline{\lambda}_j}\|+\|\sum_{n\geq N_1}^{N_2}h_n\psi_n\sum_{j\neq n}\frac{x_jb_j}{\overline{\lambda}_n-\overline{\lambda}_j}\|$$

$$\leq \sum_{n\geq N_1}^{N_2}|x_n||b_n|\|\sum_{j\neq n}\frac{h_j\psi_j}{\overline{\lambda}_n-\overline{\lambda}_j}\|+a_2\left[\sum_{n\geq N_1}^{N_2}|h_n|^2\left|\sum_{j\neq n}\frac{x_jb_j}{\overline{\lambda}_n-\overline{\lambda}_j}\right|^2\right]^{\frac{1}{2}}\leq \tilde{M}\|x\|.$$

Therefore

$$\|\sum_{n\geq N_1}^{N_2}\frac{1}{2\pi i}\oint_{C_n} R(\lambda;A^*)TR(\lambda;A^*)d\lambda\|_{\mathcal{L}(H,H)}\leq \tilde{M}. \tag{10}$$

Similar computations led to the following :

$$(R(\lambda;A^*)h,b)R(\lambda;A^*)TR(\lambda;A^*)x = \left(\frac{b_nh_n}{\lambda-\overline{\lambda}_n}+\sum_{j\neq n}^{+\infty}\frac{b_jh_j}{\lambda-\overline{\lambda}_j}\right)\left(\frac{b_nx_n}{\lambda-\overline{\lambda}_n}+\sum_{j\neq n}^{+\infty}\frac{b_jx_j}{\lambda-\overline{\lambda}_j}\right)\times$$

$$\left(\frac{h_n\psi_n}{\lambda-\overline{\lambda}_n}+\sum_{j\neq n}^{+\infty}\frac{h_j\psi_j}{\lambda-\overline{\lambda}_j}\right).$$

By the same reason, each infinite sum in the above expression converges uniformly in the disc D_n, and is analytic there. The Residue Theorem applies again :

$$\frac{1}{2\pi i}\oint_{C_n}(R(\lambda;A^*)h,b)R(\lambda;A^*)TR(\lambda;A^*)xd\lambda = -x_nb_nh_n\psi_n\sum_{j\neq n}^{+\infty}\frac{h_jb_j}{(\overline{\lambda}_n-\overline{\lambda}_j)^2}$$

$$-b_nh_n^2\psi_n\frac{x_jb_j}{(\overline{\lambda}_n-\overline{\lambda}_j)^2}+h_n\psi_n\sum_{j\neq n}^{+\infty}\frac{h_jb_j}{\overline{\lambda}_n-\overline{\lambda}_j}\sum_{k\neq n}^{+\infty}\frac{x_kb_k}{\overline{\lambda}_n-\overline{\lambda}_k}-b_n^2h_nx_n\sum_{j\neq n}^{+\infty}\frac{h_j\psi_j}{(\overline{\lambda}_n-\overline{\lambda}_j)^2}$$

$$+x_n b_n \sum_{j\neq n}^{+\infty} \frac{h_j b_j}{\overline{\lambda}_n - \overline{\lambda}_j} \sum_{k\neq n}^{+\infty} \frac{h_k \psi_k}{\overline{\lambda}_n - \overline{\lambda}_k} + b_n h_n \sum_{j\neq n}^{+\infty} \frac{x_j b_j}{\overline{\lambda}_n - \overline{\lambda}_j} \sum_{k\neq n}^{+\infty} \frac{h_k \psi_k}{\overline{\lambda}_n - \overline{\lambda}_k}.$$

Accordingly we have :

$$\sum_{n\geq N_1}^{N_2} \frac{1}{2\pi i} \oint_{C_n} (R(\lambda; A^*)h, b) R(\lambda; A^*) T R(\lambda; A^*) x d\lambda = - \sum_{n\geq N_1}^{N_2} x_n b_n h_n \psi_n \sum_{j\neq n}^{+\infty} \frac{h_j b_j}{(\overline{\lambda}_n - \overline{\lambda}_j)^2}$$

$$- \sum_{n\geq N_1}^{N_2} b_n h_n^2 \psi_n \sum_{j\neq n}^{+\infty} \frac{x_j b_j}{(\overline{\lambda}_n - \overline{\lambda}_j)^2} + \sum_{n\geq N_1}^{N_2} h_n \psi_n \sum_{j\neq n}^{+\infty} \frac{h_j b_j}{\overline{\lambda}_n - \overline{\lambda}_j} \sum_{k\neq n}^{+\infty} \frac{x_k b_k}{\overline{\lambda}_n - \overline{\lambda}_k}$$

$$- \sum_{n\geq N_1}^{N_2} b_n^2 h_n x_n \sum_{j\neq n}^{+\infty} \frac{h_j \psi_j}{(\overline{\lambda}_n - \overline{\lambda}_j)^2} + \sum_{n\geq N_1}^{N_2} x_n b_n \sum_{j\neq n}^{+\infty} \frac{h_j b_j}{\overline{\lambda}_n - \overline{\lambda}_j} \sum_{k\neq n}^{+\infty} \frac{h_k \psi_k}{\overline{\lambda}_n - \overline{\lambda}_k}$$

$$+ \sum_{n\geq N_1}^{N_2} b_n h_n \sum_{j\neq n}^{+\infty} \frac{x_j b_j}{\overline{\lambda}_n - \overline{\lambda}_j} \sum_{k\neq n}^{+\infty} \frac{h_k \psi_k}{\overline{\lambda}_n - \overline{\lambda}_k}.$$

By making tedious but elementary estimations on the above expression, one may find that

$$\| \sum_{n\geq N_1}^{N_2} \frac{1}{2\pi i} \oint_{C_n} (R(\lambda; A^*)h, b) R(\lambda; A^*) T R(\lambda; A^*) x d\lambda \| \leq \tilde{M} \|x\|,$$

or,

$$\| \sum_{n\geq N_1}^{N_2} \frac{1}{2\pi i} \oint_{C_n} (R(\lambda; A^*)h, b) R(\lambda; A^*) T R(\lambda; A^*) d\lambda \|_{\mathcal{L}(H,H)} \leq \tilde{M}. \tag{11}$$

Now let us consider the estimation of the last term in (8). Using Lemma 2, it is easy to see that for all $\lambda \in C_n$,

$$\|(R(\lambda; A^*)h, b)^2 R(\lambda; A^*) T R(\lambda; A^*) x\| = \| \left(\sum_{j=1}^{+\infty} \frac{h_j b_j}{\lambda - \overline{\lambda}_j} \right)^2 \sum_{k=1}^{+\infty} \frac{x_k b_k}{\lambda - \overline{\lambda}_k} \sum_{l=1}^{+\infty} \frac{h_l \psi_l}{\lambda - \overline{\lambda}_l} \|$$

$$\leq \left| \sum_{j=1}^{+\infty} \frac{h_j b_j}{\lambda - \overline{\lambda}_j} \right|^2 \left| \sum_{k=1}^{+\infty} \frac{x_k b_k}{\lambda - \overline{\lambda}_k} \right| a_2 \left(\sum_{l=1}^{+\infty} \left| \frac{h_l}{\lambda - \overline{\lambda}_l} \right|^2 \right)^{\frac{1}{2}} \leq \tilde{M} \left(\frac{1}{d_n} \right)^{3(1-\tau)} \left(\frac{1}{d_n} \right) \|x\|.$$

Equivalently,

$$\|(R(\lambda; A^*)h, b)^2 R(\lambda; A^*) T R(\lambda; A^*)\|_{\mathcal{L}(H,H)} \leq \tilde{M} \left(\frac{1}{d_n} \right)^{3(1-\tau)} \left(\frac{1}{d_n} \right). \tag{12}$$

Using (12), (6) and H3 (ii), the estimation of the last term of (8) can be carried out for N_1 sufficiently large as follows :

$$\| \sum_{n\geq N_1} \frac{1}{2\pi i} \oint_{C_n} \frac{(R(\lambda; A^*)h, b)^2 R(\lambda; A^*) T R(\lambda; A^*)}{1 - (R(\lambda; A^*)h, b)} d\lambda \|_{\mathcal{L}(H,H)}$$

$$\leq \| \sum_{n\geq N1} \frac{1}{2\pi i} \oint_{C_n} \frac{(R(\lambda;A^*)h,b)^2 R(\lambda;A^*)TR(\lambda;A^*)}{1-(R(\lambda;A^*)h,b)} d\lambda \|_{\mathcal{L}(H,H)}$$

$$\leq \sum_{n\geq N1} d_n \sup_{\lambda\in C_n} \|(R(\lambda;A^*)h,b)^2 R(\lambda;A^*)TR(\lambda;A^*)\|_{\mathcal{L}(H,H)} \sup_{\lambda\in C_n} |\frac{1}{1-(R(\lambda;A^*)h,b)}|$$

$$\leq \tilde{M}. \tag{13}$$

From (8), (10), (11) and (13), it is easy to get

$$\| \sum_{n\geq N1}^{N_2} [E'_n - E(\lambda_n)] \|_{\mathcal{L}(H,H)} \leq 3\tilde{M}, \tag{14}$$

where $\tilde{M}$ is independent of N_2. Since $\sum_{n\in\mathbb{N}} E(\lambda_n) = I$ and every compact in the complex plane contains only a finite number of elements of $\sigma(L_h)$, it follows from (14) that $\| \sum_{\lambda\in\sigma(L_h)} E'_\lambda \|_{\mathcal{L}(H,H)}$ is bounded. In other words, we have proved that L_h is regular spectral.

Repeating the same estimations (8), (10), (11) and (13) with $N_2 = N_1$, it is not difficult to see that for some fixed N_1 large enough, $\|E'_n - E(\lambda_n)\| < 1$ for all $n \geq N_1$. Therefore the two projections have the same dimension (see p.33 [6]). From H1, we know that E'_n is one-dimensional, that is there is only one spectrum point $\overline{v}_n$ in each D_n. This proves a).

To prove that

$$\sum_{\lambda\in\sigma(L_h)} E'_\lambda = I,$$

the argumentation of [7] is adapted here to our case by using the Lemma 16 of [14] which says that $\sum_{\lambda\in\sigma(L_h)} E'(\lambda) = I$ if and only if the following set $S_\infty(L_h)$ is reduced to $\{0\}$:

$$S_\infty(L_h) = \{f; g(\lambda) = R(\lambda; L_h)f \text{ is an entire function }\}.$$

Construct a closed path Γ such that $\Gamma \bigcap (\bigcup_{n\geq 1} D_n \bigcup \{\lambda; |\lambda| \leq M_1\}) = \emptyset$. From (6), it is known that for some fixed M_1 large enough, $\lambda \in \Gamma$ implies that $\|TR(\lambda;A^*)\|_{\mathcal{L}(H,H)} \leq \frac{1}{2}$. Consider now an arbitrary element $f \in S_\infty(L_h)$. Then $\lambda \in \Gamma$ implies that

$$\|R(\lambda;L_h)f\| \leq \|R(\lambda;A^*)(1-TR(\lambda;A^*))^{-1}f\|$$

$$\leq \|R(\lambda;A^*)\|_{\mathcal{L}(H,H)} \|(1-TR(\lambda;A^*))^{-1}\|_{\mathcal{L}(H,H)} \|f\|$$

$$\leq 2\|R(\lambda;A^*)\|_{\mathcal{L}(H,H)}\|f\| \leq 2\sup_{k\in\mathbb{N}} \frac{1}{|\lambda-\lambda_k|}\|f\|.$$

Since $R(\lambda;L_h)f$ is an entire function, by virtue of the maximum principle, we know that for all λ in the region bordered by Γ,

$$\|R(\lambda;L_h)f\| \leq 2\|f\| \sup_{\mu\in\Gamma} \sup_{k\in\mathbb{N}} \frac{1}{|\mu-\lambda_k|}.$$

In particular, for some fixed $\tilde{\lambda} \in \rho(L_h)$ in the bordered region, we have

$$\|R(\tilde{\lambda}; L_h)f\| \leq 2\|f\| \sup_{\mu\in\Gamma} \sup_{k\in\mathbb{N}} \frac{1}{|\mu - \lambda_k|}. \tag{15}$$

It is easy to see that by increasing M_1 the left side of (15) can be made as small as we like. Hence $R(\tilde{\lambda}; L_h)f = 0$. Therefore $f = 0$ because of $\tilde{\lambda} \in \rho(L_h)$. f being arbitrary implies that $S_\infty(L_h) = \{0\}$. So we prove Theorem 1. □

REMARK : If the eigenvalues $\sigma(A^*)$ are simple, those of the perturbed operator L_h are also simple excepted a finite number of them. As shown in the proof of Theorem 1, for a fixed $h \in H$, every point of $\sigma(L_h)$, but a finite number of them, is located in the corresponding D_n. By refining the radius of D_n we get the direct generalization of the result of [21].

THEOREM 2 : *Assume that the hypothesis H1, H2 and H3 are verified and that* $h = \sum_{k=1}^{+\infty} h_k\psi_k \in H$. *Then for all n large enough,*

$$|\overline{v}_n - \overline{\lambda}_n| \leq 4|b_n h_n|,$$

where $\overline{v}_n \in \sigma(L_h)$.

Proof : If $h_n = 0$, from Proposition 1 $v_n = \lambda_n$. Hence the above inequality is true. Suppose now that $h_n \neq 0$ for all $n \in \mathbb{N}$ without loosing the generality. Consider the discs $\tilde{D}_n$ defined similarly as previously : $\tilde{D}_n = \{\lambda; |\lambda - \overline{\lambda}_n| \leq 4|b_n h_n|\}$. From H3 (i) and the fact that $\sum_{n=1}^{+\infty} |h_n|^2 < +\infty$, it is clear that $4|b_n h_n| \leq \frac{1}{3}d_n$, or $\tilde{D}_n \subset D_n$ for all n large enough. By virtue of Theorem 1, it is sufficient to show that for all n large enough, $D_n \backslash \tilde{D}_n \subset \rho(L_h)$. Indeed from Lemma 2, it is true that for all $\lambda \in D_n \backslash \tilde{D}_n$,

$$|(R(\lambda; A^*)h, b)| \leq \sum_{j\neq n}^{+\infty} \left| \frac{b_j h_j}{\lambda - \overline{\lambda}_j} \right| + \frac{1}{4} \leq K\|h\| \left(\frac{1}{d_n}\right)^{1-\tau} + \frac{1}{4}.$$

From H3 (ii) there exists a N_1 so large that for all $n \geq N_1$, $K\|h\| \left(\frac{1}{d_n}\right)^{1-\tau} \leq \frac{1}{4}$ and $\tilde{D}_n \subset D_n$. Therefore $|(R(\lambda; A^*)h, b)| \leq \frac{1}{2}$. Direct application of Proposition 1 proves Theorem 2. □

3 SPECTRUM ASSIGNMENT RESULTS

Let $\{v_k; k \in \mathbb{N}\}$ be the spectrum to be assigned and the real part of the spectrum : $\omega_c = \sup\{\Re e(v_k); k \in \mathbb{N}\}$ be bounded. Otherwise the operator A_h is not an infinitesimal generator of C_0−semigroup on H. Now what are the limitations to the spectrum assignment when we have only BLF? From the last section, we have immediately :

COROLLARY 1 : *If $\{v_k; k \in \mathbb{N}\}$ is the real-part-bounded spectrum of A_h assigned via BLF, then*

$$(SA) \quad : \qquad \sum_{k=1}^{+\infty} \left| \frac{v_k - \lambda_k}{b_k} \right|^2 < +\infty.$$

Moreover A_h (resp. L_h) is an infinitesimal generator of C_0-semigroup $S_h(t)$ (resp. $S_h^*(t)$) on H which satisfies the spectrum determined growth assumption :

$$\|S_h(t)\|_{\mathcal{L}(H,H)} \leq K e^{t\omega_c} \textit{ for some constant } K.$$

<u>Proof</u> : The first part is a direct consequence of Theorem 2 and the second one is classical [7]. □

Now we prove that the condition (SA) is sufficient, and we explicitly give the BLF for the spectrum assignment problem.

THEOREM 3 : *Let the set $\{v_k; k \in \mathbb{N}\}$ (which is assumed to be simple such that $v_k \neq \lambda_j$ for all $k, j \in \mathbb{N}$.) satisfy the necessary condition of Corollary 1. The spectrum $\sigma(A_h)$ of the closed loop system (Σ_C) is assigned at $\{v_k\}$ with the unique element $h = \sum_{k=1}^{+\infty} h_k \psi_k \in H$ which is the solution of the following algebric equation :*

$$\sum_{k=1}^{+\infty} \frac{\overline{v}_m - \overline{\lambda}_m}{b_m} \frac{b_k h_k}{\overline{v}_m - \overline{\lambda}_k} = \frac{\overline{v}_m - \overline{\lambda}_m}{b_m}; \; m \in \mathbb{N}. \tag{16}$$

<u>Proof</u> : We will give an explicit construction of the unique solution of (16) and prove that for all $n \in \mathbb{N}$, $h_n \neq 0$. On the other hand, we will prove that $\Lambda = \{\overline{v}_m\}_{m \in \mathbb{N}}$. It will follow from Proposition 1 that $\sigma(L_h) = \{\overline{v}_m; m \in \mathbb{N}\}$. Thus $\sigma(A_h) = \{v_m; m \in \mathbb{N}\}$.

We consider the equation (16) in l^2 as follows : $(1+T)h = g$ where $g = \left(\frac{\overline{v}_m - \overline{\lambda}_m}{b_m}\right)_{m \in \mathbb{N}} \in l^2$ and $T_{m,n} = \frac{\overline{v}_m - \overline{\lambda}_m}{b_m} \frac{b_n}{\overline{v}_m - \overline{\lambda}_n}$ for $m \neq n$ and $T_{n,n} = 0$. The solution h of (16) as shown in [21] is unique because the infinite product $\prod_{j=1}^{+\infty} \prod_{k=j+1}^{+\infty} \frac{(\overline{v}_k - \overline{v}_j)(\overline{\lambda}_j - \overline{\lambda}_k)}{(\lambda_k - \overline{v}_j)(\lambda_j - \overline{v}_k)}$ still converges. We prove here that the unique solution h is exactely the limit of the sequence $\tilde{h}^n$ in l^2 which is the solution of the following equation :

$$(1 + \tilde{T}^n)\tilde{h}^n = \tilde{g}^n \tag{17}$$

where $\tilde{h}^n = (\tilde{h}_1^n, ..., \tilde{h}_2^n, 0, ..., 0, ...)^t$, $\tilde{g}^n = (\dfrac{\overline{v}_1 - \overline{\lambda}_1}{b_1}, ..., \dfrac{\overline{v}_n - \overline{\lambda}_n}{b_n}, 0, ..., 0, ...)^t$

and $\tilde{T}_{i,j}^n = T_{i,j}$ for $i \leq n$ and $j \leq n$ and $\tilde{T}_{i,j}^n = 0$ elsewhere.

By direct matrix computations, we can show that the unique solution of (17) is

$$\tilde{h}_k^n = \frac{\overline{v}_k - \overline{\lambda}_k}{b_k} \prod_{j \neq k}^{n} \left(\frac{\overline{v}_j - \overline{\lambda}_j}{\overline{\lambda}_j - \overline{\lambda}_k} + 1 \right) \quad \text{for } 1 \leq k \leq n \quad \text{and } \tilde{h}_k^n = 0 \ \text{ for} k \geq n+1.$$

Since the infinte product in the above expression converges uniformly for $n \to +\infty$, $\tilde{h}^n$ is a Cauchy sequence in l^2 and so converges to h^* for $n \to +\infty$. From the fact that $\sum_{k=1}^{+\infty} \sum_{j \neq k}^{+\infty} \left| \frac{v_k - \lambda_k}{b_k} \right|^2 \left| \frac{b_j}{v_k - \lambda_j} \right|^2 < +\infty$, $\|T - \tilde{T}^n\|_{\mathcal{L}(l^2, l^2)} \overset{n \to +\infty}{\longrightarrow} 0$. Then $(1+T)h^* - g = \lim_{n \to +\infty} \{h^* - \tilde{h}^n + T(h - \tilde{h}^n) + (T - \tilde{T}^n)\tilde{h}^n + g_n - g\} = 0$. The uniqueness of the solution of (16) tells us that $h = h^*$.

Now suppose that for some $m \in \mathbb{N}$, $h_m = 0$. Direct computation leads to $\|\tilde{h}^n - h\|_{l^2} \geq |\tilde{h}_m^n - h_m| = \left| \frac{v_m - \lambda_m}{b_m} \right| \left| \prod_{j=1, j \neq m}^{n} \frac{\overline{v}_j - \overline{\lambda}_m}{\lambda_j - \lambda_m} \right| > 0$. This is contradictory with the fact that $\lim_{n \to +\infty} \|\tilde{h}^n - h\|_{l^2} = 0$. Therefore $h_n \neq 0$ for all $n \in \mathbb{N}$. From Proposition 1, we have :

$$\sigma(L_h) = \Lambda. \tag{18}$$

By construction, $\sigma(L_h) = \Lambda \supset \{\overline{v}_k; k \in \mathbb{N}\}$ because $\Delta_h(\overline{v}_k) = 0$ (see (16)). We have to prove that $\sigma(L_h) = \{\overline{v}_k; k \in \mathbb{N}\}$. From Theorem 1, the spectrum points of L_h from certain rank N_0 are contained in $\bigcup_{n > N_0} D_n$ and each D_n contains only one point of $\sigma(L_h)$. From our explicit construction and the relation (18) we have that

$$\{\sigma_k; \sigma_k \in \Lambda, k > N_0\} = \{\overline{v}_k; k > N_0\}.$$

So we have only to prove that

$$\{\sigma_k; \sigma_k \in \Lambda, 1 \leq k \leq N_0\} = \{\overline{v}_k; 1 \leq k \leq N_0\}.$$

Now consider one compact which excludes the set $\{\overline{v}_k; k > N_0\}$ and the following analytic functions on this compact :

$$F(\lambda) = \Delta_h(\lambda) \prod_{j=1}^{N_0} (\lambda - \overline{\lambda}_j) \quad \text{and} \quad F_n(\lambda) = \Delta_{\tilde{h}^n}(\lambda) \prod_{j=1}^{N_0} (\lambda - \overline{\lambda}_j).$$

The function sequence $F_n(\lambda)$ converges uniformly to $F(\lambda)$ on the compact because for some constant M_2, $|F_n(\lambda) - F(\lambda)| \leq M_2 [\|\tilde{h}^n - h\|_{l^2} + (\sum_{k=n+1}^{+\infty} |h_k|^2)^{\frac{1}{2}}] \overset{n \to +\infty}{\longrightarrow} 0$. Moreover $F_n(\lambda) = \prod_{j=1}^{N_0} (\lambda - \overline{v}_j) \prod_{k=N_0+1}^{n} \left(\frac{\lambda - \overline{v}_k}{\lambda - \overline{\lambda}_k} \right)$. It follows from Th. 15.4 of [19] (p.291 [19]) that the only zeros of $F(\lambda)$ in the compact are $\{\overline{v}_k; k \leq N_0\}$. Hence $\Lambda = \{\overline{v}_k; k \in \mathbb{N}\}$. This proves Theorem 3. □

REMARK : The proof of Theorem 3 gives also a systematic method to assign a finite number of points of $\sigma(A)$ so that $\sigma(A_h) = \{v_k; k \leq n\} \bigcup \{\lambda_k; k \geq n+1\}$. A natural practical application of this method was presented in [18]. The previous results give some consequences on the stabilizability of abstract vibration equations via BLF. Firstly let $\sigma(A) = \{i\lambda_k; \lambda_k \in \mathbb{R},\ k \in \mathbb{N}\}$ and $b_k = O(k^\alpha)$ for $k \in \mathbb{N}$ with $\alpha > \frac{1}{2}$. If A and b satisfy the conditions H1, H2 and H3, there exists a unique BLF which assigns the spetrum $\sigma(A_h)$ at $\{i\lambda_k - \epsilon^2; k \in \mathbb{N}\}$ with ϵ constant. In this case, one can assign uniformly an infinite number of eigenvalues so that the closed loop system is exponentially stable. We can construct many examples as this one for the diagonal semigroup on l^2 [22]. However $b_n = O(n^\alpha)$ $(\alpha > \frac{1}{2})$ is not admissible in the sense of [11] (see [22][8]). Under H1, H2 and H3, b is admissible if and only if $\{b_k\} \in l^\infty$ [22].

Then we get the following negative consequence for the stabilizability :

COROLLARY 2 : *Assume that the hypothesis H1, H2 and H3 are verified and* $\sigma(A) = \{i\lambda_k; \lambda_k \in \mathbb{R},\ k \in \mathbb{N}\}$. *Then the following two assertion are equivalent :*

1) If b is admissible, it is impossible to exponentially stabilize the system by any BLF.

2) If there exists some BLF which stabilizes exponentially the system, then b is not admissible.

Proof : b being admissible implies that $\{b_k\} \in l^\infty$. From Corollary 1 it is necessary that $\lim_{k\to+\infty} |v_k - \lambda_k| = 0$. This proves 1). The equivalence between 1) and 2) is evident. □

REMARK : In Corollary 2, although we do not obtain the exponential stability with BLF when b is admissible, we can make the closed loop system strongly stable with it.

To illustrate how to apply our theoretical results, we consider the stabilization problem for the rotating body-beam model given in [1]. It was proved in [17] that with only torque control, it is impossible to stabilize this sytem at the angular velocity greater than the critical one. However the stabilization of the system at any constant angular velocity can be realized by adding a lateral force control (see [18]). Essentially, the lateral force control effect may be viewed via the following example :

EXAMPLE 1 :

$$\begin{aligned} \frac{\partial u_1(x,t)}{\partial t} &= u_2(x,t) \\ \frac{\partial u_2(x,t)}{\partial t} &= -\frac{\partial^4 u_1(x,t)}{\partial x^4} - K u_2(x,t) + \omega^{*2} u_1(x,t) \end{aligned}$$

with boundary conditions: $u_1(0,t) = u_{1x}(0,t) = u_{1xx}(1,t) = 0$ and $u_{1xxx}(1,t) = F(t)$, where $K > 0$ signifies the presence of viscous friction and ω^* is the constant angular velocity on which the system is to be stabilized. Take $H = H_0^2(0,1) \times L^2(0,1)$ for the phase space where $H_0^2(0,1) = \{f; f, f', f'' \in L^2(0,1), f(0) = f'(0) = 0\}$ and the inner product on H is $< f,g >= \int_0^1 [f_{1xx}(x) g_{1xx}(x) + f_2(x) g_2(x)]dx$. It is clear that $A = \begin{pmatrix} 0 & 1 \\ -\frac{\partial^4}{\partial x^4} + \omega^{*2} & -K \end{pmatrix}$ with domain

$$\mathcal{D}(A) = \left\{ f = (f_1, f_2)^t; f \in H,\ f_1 \in H^4,\ f_{1xx}(1) = 0,\ f_{1xxx}(1) = 0,\ f_2 \in H_0^2 \right\}$$

is an infinitesimal generator of C_0-semigroup on H and $b = -(0, \delta(1-\cdot))^t$ (δ is Dirac function). By direct computations we found that

$$\sigma(A^*) = \left\{ \overline{\lambda}_n; \lambda_n = \frac{1}{2}[-K + sign(n)\sqrt{K^2 - 4\mu_{|n|} + 4\omega^{*2}}], n = \pm1, \pm2, ... \right\}$$

$$\psi_n(x) = \frac{\sqrt{\mu_{|n|} + |\lambda_n|^2}}{K + 2\lambda_n} \begin{pmatrix} \frac{K + \overline{\lambda}_n}{\mu_{|n|}} \\ 1 \end{pmatrix} e_{|n|}(x),$$

where μ_n and e_n are respectively the eigenvalue and the eigenvector of the unbounded operator $\frac{\partial^4}{\partial x^4}$ in $L^2(0,1)$ with

$$\mathcal{D}(\frac{\partial^4}{\partial x^4}) = \{f; f, f, f'' \in L^2(0,1), f(0) = f'(0) = f''(1) = f'''(1) = 0\}.$$

It is easy to see that $\mu_n = \left[n\pi + \frac{\pi}{2} + O(e^{-n\pi})\right]^4$ for all $n \in \mathbb{N}$ and $0 < m_1 \leq |b_n| \leq m_2$. Therefore A and b can be verified to satisfy H1,H2 and H3.

THEOREM 4 : *Given any constant ω^*, there exists a positive integer N_0 and $H \ni h = \sum_{j=1}^{N_0} [h_j \psi_j + h_{-j} \psi_{-j}]$ such that $F(t) =< u(\cdot, t), h >$ stabilizes exponentially the system.*

Proof : Given any constant ω^*, $\sigma(A^*)$ has only a finite number of points in $\Re e(s) \geq 0$ because $\mu_n \overset{n \to +\infty}{\longrightarrow} +\infty$. These points are displaced to $\Re e(s) < 0$ by the computed BLF. Thus the rest of the proof follows directly from Theorem 3 and Corollary 1. □

4 CONCLUSIONS, POSSIBLE DEVELOPMENTS

In this paper, the necessary and sufficient condition of Sun [21] was generalized in somes cases of boundary control, which allows to exploit the limitations imposed by BLF and that by admissibility of the input vector. We have proved that it is possible to achieve exponential stabilization of some abstract vibration systems by BLF. Indirectly the paper gived also a systematic method to assign a finite number of spetrum points. This method could find potential applications in damped flexible systems as illustrated by our example (see [13] for other models). We are led to believe that our result could be more generalized to replace the condition H3 (ii) by the following one (see the proof of Th.1) :

$$\sum_{n=1}^{+\infty} \left(\frac{1}{d_n}\right)^{m_0(1-\tau)} < +\infty \text{ for some positive integer } m_0 \geq 3.$$

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